16p.

UNPUBLISHED PRELIMINARY DATA

MH MPG Report 1541-TR 8

7 January 1964

Honeywell

N64-23851

CODE-1 CAT. 20

NASA CR-56650

Downtown

ROCKET BOOSTER CONTROL

SECTION 8

LINEAR PROGRAMMING

AND

BOUNDED PHASE COORDINATE CONTROL

NASA Contract NASw-563

OTS PRICE

XEROX $ 1.60 ph

MICROFILM $ none

REC

MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT

cr

MH MPG Report 1541-TR 8 7 January 1964

ROCKET BOOSTER CONTROL

SECTION 8

LINEAR PROGRAMMING

AND

BOUNDED PHASE COORDINATE CONTROL

NASA Contract NASw-563

Prepared by: D. L. Russell

D. L. Russell
Research Consultant

Supervised by: C. R. Stone

C. R. Stone
Research Supervisor

Approved by: O. H. Schuck

O. H. Schuck
Director
MPG Research

HONEYWELL
MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT
Minneapolis, Minnesota

FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

TABLE OF CONTENTS

LINEAR PROGRAMMING AND BOUNDED PHASE COORDINATE CONTROL*

by D. L. Russell‡

23851

ABSTRACT

A maximum principle for time-optimal bounded-phase coordinate control of ordinary linear differential equation systems and an existence theorem are proven. Plant differential equations are then approximated by recurrence equations so the problem of obtaining "optimal" trajectories by backing out of the origin can be reduced to one of linear programming.

author

INTRODUCTION

The plant considered is represented by the vector differential equation

$$\dot{x} = A(t)x + B(t)u + f(t) \tag{1}$$

where x and u are n-and m-vectors, respectively; A(t) and B(t) are continuous n x n and n x m matrices; and f(t) is an n-vector. The vector x is called the phase variable; u the control variable. It is assumed that f(t) is measurable on some interval I and bounded on every compact subinterval of I; that u(t) is measurable on various subintervals $t_o \leq t \leq t_1$ in I (which are the domains of definition of the u's); and that u(t) lies in a non-empty,

* Prepared under contract NASw-563 for the NASA.

‡ Research Consultant, Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota

compact, convex restraint set $\Omega \subset R^m$ which contains more than one point. Trajectories corresponding to u(t) are denoted by x(t).

G is a closed, convex, non-empty subset of R^n. The initial point $x(t_o) = x_o \in G$. T is a closed, non-empty target set and $T \subset G$. It is assumed $x_o \notin T$. Δ is the set of all control functions u such that $x(t_o) = x_o$ and $x(t_1) \in T$ and $x(t) \in G$ for $t_o \leq t \leq t_1$. The objective is to determine a control function $u^*(t) \in \Delta$ defined on $[t_o^*, t_1^*]$ such that $t_1^* - t_o^* \leq t_1 - t_o$ for every other $u \in \Delta$.

The method developed herein permits trajectories to be determined by backing out of the origin. The plant differential equations (1) are approximated by recurrence equations. The control variables are constant over sample intervals and the phase bounds are only enforced at the endpoints of these intervals. The trajectories thus determined are approximations to the desired optimal trajectories.

Theorem 1 below shows that optimal trajectories are extremal in the sense of Definition 1. Then the objectives for existence of extremal trajectories is established. The plant differential equations are replaced by recurrence equations and the problem solution is reduced to one of linear programming.

THE MAXIMUM PRINCIPLE

The maximum principle is a statement that controls that displace the plant (equation 1) the maximum distance in a given direction from an initial point during a fixed time interval are necessarily extremal controls; the extremal controls being defined beforehand.

It is first assumed the initial time, t_o, is fixed for all controls, u. The set of attainability, $K(t,x_o)$ is then defined as the set of $x(t)$ corresponding to controllers $u(\tau)$ defined on $t_o \leq \tau \leq t$ with $u(\tau) \in \Omega$ and such that $x(\tau) \in G$ for $t_o \leq \tau \leq t$. $K(t,x_o)$ is closed, bounded, convex, and varies continuously with t.

It is clear that the smallest number t_1 with $K(t_1,x_o) \cap T \neq \phi$ is the minimal time required to bring x_o into T. Each optimal control $u^*(t)$ has the property that $x(t_1^*)$ lies on the boundary of $K(t_1^*,x_o)$.

The adjoint system for the system (1) is

$$\dot{\psi}' = -\psi' A(t) \tag{2}$$

where ψ' indicates the transpose of ψ.

<u>DEFINITION 1.</u> A control $u^*(t)$ defined on $[t_o,t_1^*]$ and lying in Ω with $x(t)$ lying in G on $[t_o,t_1^*]$ is called extremal if there exists a solution ψ' of (2) defined on $[t_o,t_1^*]$ such that

$$\int_{t_o}^{t_1^*} \psi'(s)\, B(s)\, u^*(s)ds \geq \int_{t_o}^{t_1^*} \psi'(s)\, B(s)\, u(s)ds$$

for all other $u(t)$ lying in Ω with $x(t)$ lying in G on $[t_o,t_1^*]$.

<u>THEOREM 1.</u> If the response $x^*(t)$ to the control $u^*(t)$ defined on $[t_o,t_1^*]$ is such that $x^*(t_1^*) \in \partial K(t_1^*,x_o)$, then the controller $u^*(t)$ is extremal.

<u>PROOF:</u> The assumptions that G and Ω are convex implies that $K(t_1^*, x_o)$ is convex. Thus $K(t_1^*, x_o)$ may be supported by a plane π at $x^*(t_1^*)$. ψ_1 is taken as the unit outward normal vector to π.

Then it is clear by convexity of $K(t_1^*, x_o)$ that for every point $Q \in K(t_1^*, x_o)$,

$$\psi_1' x^*(t_1^*) \geq \psi_1' Q \tag{3}$$

From (1)

$$x^*(t_1^*) = \Phi(t_1^*)x_o + \Phi(t_1^*) \int_{t_o}^{t_1^*} \Phi^{-1}(s)[B(s)u^*(s) + f(s)]ds \tag{4}$$

where $\Phi(t)$ is the fundamental solution matrix of the homogeneous equation corresponding to (1) with $\Phi(t_o) = I$. $\psi'(t)$ is taken as the solution of (2) such that $\psi'(t_1^*) = \psi_1'$. Then (3) is rewritten

$$\begin{aligned} &\psi'(t_1^*)\ x^*(t_1^*) = \psi'(t_1^*)\, \Phi(t_1^*)x_o + \\ &\psi'(t_1^*)\, \Phi(t_1^*) \int_{t_o}^{t_1^*} \Phi(s)^{-1}[B(s)u^*(s) + f(s)]ds \geq \\ &\psi'(t_1^*)\ \Phi(t_1^*)x_o + \psi'(t_1^*)\, \Phi(t_1^*) \int_{t_1}^{t_1^*} \Phi(s)^{-1}[B(s)u(s) + f(s)]ds \end{aligned} \tag{5}$$

for all other controllers $u(t)$ defined on $[t_o, t_1^*]$ such that $u(t)$ lies in Ω and $x(t)$ lies in G.

By writing $\psi'(t) = \psi_o'\Phi(t)^{-1}$ for an appropriate ψ_o', it is found that

$$\int_{t_o}^{t_1^*} \psi'(s)\ B(s)\ u^*(s)ds \geq \int_{t_o}^{t_1^*} \psi'(s)\ B(s)\ u(s)ds \tag{6}$$

which completes the proof.

It is noted that in this case it may not in general be concluded that $\psi'(s)\ B(s)\ u^*(s) \geq \psi'(s)\ B(s)\ u(s)$ for almost all $s \in [t_o, t_1^*]$. This is possible in the case where $G = R^n$.

BACKING OUT OF THE ORIGIN

If families of optimal trajectories could be obtained, it would be expected the optimal control could be synthesized. If the expressions for the optimal trajectories were simple enough, explicit expressions for the controls could be developed. Otherwise the trajectory data could be calculated and placed in a storage computer.

The backing out of the origin procedure consists of determining optimal trajectories for the system (1) by starting with zero initial conditions and figuratively running the system backwards. It is assumed the target T is the origin, the constraint set G is a polyhedron in R^m, and the system (1) is autonomous; that is, A, B, and f are constant.

Basically the method may be described as follows. ψ' is taken as a solution of (2) defined on $[t_o, t_1]$. Then $u^*(t)$ is found on $[t_o, t_1]$ so that

$$\int_{t_o}^{t_1} \psi'(s)\ Bu^*(s)ds \geq \int_{t_o}^{t_1} \psi'(s)\ Bu(s)ds, u^*(t) \text{ and } u(t)$$

lying in Ω and subject to the condition that the solution of (1) with $x(t_1) = 0$ lies always in G on $[t_o, t_1]$.

If it can be shown that $u^*(t)$ uniquely maximizes $\int_{t_o}^{t_1} \psi'(s)Bu(s)ds$ then the response $x(t)$ must be optimal carrying $x(t_o)$ into the origin at time t_1. Each such ψ' thus determines an optimal response $x(t)$.

A method for maximizing $\int_{t_o}^{t_1} \psi'(s)Bu(s)ds$ to any desired degree of accuracy, while fulfilling both the constraints on the control $u(t)$ and the phase variable $x(t)$ is presented below.

The existence of a $u^*(t)$ maximizing $\int_{t_o}^{t_1} \psi'(s)Bu(s)ds$ and fulfilling the constraint requirements on $u(t)$ and $x(t)$ is established first.

EXISTENCE OF A MAXIMIZING $u^*(t)$

$\psi'(t)$ is taken on the finite interval $[t_o,t_1]$ as a solution of (2). For a given $u(t)$ defined, measurable, and bounded on the same interval, the response $x(t)$ with $x(t_1) = 0$ is given by the variation of parameters formula.

$$x(t) = \int_{t_1}^{t} e^{-A(s-t)}[Bu(s) + f]ds \tag{7}$$

where it is noted that $t \leq t_1$.

This leads to the following theorem:

<u>THEOREM 2.</u> Let $\Phi(s)$ be an m-dimensional vector in $L^2[0,T]$. Then there exists an m-dimensional vector $u^*(s) \in L^2[0,T]$ such that

1. $u^*(s)$ belongs to the class F of m-vector functions $u(s)$ in $L^2[0,T]$ for which

   (a) $u_{oj} \leq u_j(s) \leq u_{1j}$ uniformly on $[0,T]$.

   (b) For a matrix $B(s)$, measurable and bounded on $[0,T]$ of dimensions $n \times m$ and a measurable, bounded n-vector $f(s)$ defined on $[0,T]$

$$x(s) = \int_0^s [B(\sigma)u(\sigma) + f(\sigma)]d\sigma \tag{8}$$

satisfies $x(s) \in G$, $s \in [0,T]$, where G is a closed convex subset of R^n such that $0 \in G$.

2. $\int_0^T \Phi'(s)\, u^*(s)ds \geq \int_0^T \Phi'(s)u(s)ds$ for all other $u(t) \in F$.

SKETCH OF PROOF: The following properties of the set $F \subseteq L^2[0,T]$ may be easily verified:

(a) F is bounded.

(b) F is closed.

(c) F is convex.

Now set $L(u) = \int_0^T \Phi'(s)u(s)ds$. Then L is a linear functional on $L^2[0,T]$. It will now be shown that L is bounded and hence continuous. For, using the Schwartz inequality

$$|L(u)| = \left|\int_0^T \Phi\ (s)u(s)ds\right| \leq \sqrt{\int_0^T |\Phi(s)|^2\ ds} \cdot \sqrt{\int_0^T |u(s)|^2 ds} = \|\Phi\| \|u\| \tag{9}$$

Here $|\Phi(s)|$, $|u(s)|$ denote the norms of $\Phi(s)$, $u(s)$ at a given s as elements of a finite dimensional Euclidean space whereas $\|\phi\|$, $\|u\|$ are the norms of these vector functions as elements of $L^2\ [0,T]$. Since $\|\Phi\|$ is finite by hypothesis, the result that L is a bounded linear functional has been obtained.

Since $\|u\|$ is bounded in F, L(u) is bounded above in F. Since L(u) is bounded above it must have a least upper bound M. Let u_i, $i = 1,2,\ldots$ be a sequence of functions in F such that $\lim_{i \to \infty} L(u_i) = M$.

Now a closed, bounded sphere in $L^2\ [0,t]$ is weakly compact. Hence there is a subsequence of the u_i, say u_{i_k}, and an element $u^* \in L^2[0,T]$ such that $\lim_{k \to \infty} L(u_{i_k}) = L(u^*)$. Hence $L(u^*) = M$. This may be concluded because F is bounded. F is not itself necessarily a sphere in $L^2[0,T]$, but it is contained in a closed bounded sphere of $L^2[0,T]$. Thus u* exists but it is not known yet that it lies in F. To show that u* does lie in F the reader is referred to page 422 of reference 2 where the following theorem is stated and indications of the proof are given:

"A convex subset of a locally convex linear topological space is $\mathfrak{X}^*$ closed (i.e., closed in the weak topology) if and only if it is closed. (i.e., closed in the usual topology.) Since $L^2\ [0,T]$ is a locally convex linear topological space it may be concluded that $u^* \in F$. Hence $L(u^*) = M$ and hence by definition of M

$$\int_0^T \Phi'(s)u^*(s)ds \geq \int_0^T \Phi'(s)u(s)ds \tag{10}$$

for any $u \in F$. Q.E.D.

Thus the proof of the existence of the maximizing u* has been obtained. It is reasonable to suspect that u* is unique for almost all Φ in some sense.

A METHOD FOR APPROXIMATING u* FOR A GIVEN ψ

Attention is again restricted to autonomous systems.

It is now supposed that G is a convex polyhedron given by

$$V_\ell \cdot x \leq C_\ell, \ \ell = 1,2,\ldots, L, \tag{11}$$

the V_ℓ being n dimensional vectors. It is required that the restraint set Ω be given by

$$u_{oi} \leq u_i \leq u_{1i}, \ i = 1,2,\ldots,m. \tag{12}$$

A function ψ' is given on $[t_o,t_1]$ with some final conditions on $\psi'(t_1)$, as a solution of (2).

The approximation method basically consists of the assumption that u is constant on the intervals $t_o = \tau_o \leq \tau < \tau_1$, $\tau_1 \leq \tau < \tau_2, \ldots \tau_{n-1} \leq \tau < \tau_\nu = t_1$, where $\tau_j - \tau_{j-1} = \lambda_j$. Of course $\sum_{j=1}^{\nu} \lambda_j = t_1 - t_o$. Then it is seen that, if u_j is the value of $u(s)$ on $\tau_{j-1} \leq \tau < \tau_j$,

$$\int_{t_o}^{t_1} \psi'(s)Bu(s)ds = \sum_{j=1}^{\nu} \left(\int_{\tau_{j-1}}^{\tau_j} \psi'(s)B \cdot u_j \right) \equiv \Gamma_\psi \cdot U, \tag{13}$$

(U consists of treating the ν vectors u_j, each of n components, as one νm dimensional vector. Γ_ψ is similarly defined.)

Now the response to this control u is given by

$$x(t) = - \int_t^{t_1} e^{-A(s-t)}[Bu(s) + f]ds \tag{14}$$

It is desired that this response should not leave the region G. Instead of imposing this condition at every t it is imposed, for ease of computation, at each of the times $\tau_j, \tau_1, \ldots \tau_{\nu-1}$. It is automatically satisfied at $\tau_\nu = t_1$. For each τ_j, $j = 0,1,2,\ldots,\nu-1$,

$$\begin{aligned} x(\tau_j) &= - \int_{\tau_j}^{\tau_\nu = t_1} e^{-A(s-t)}[Bu(s) + f]ds \\ &= - \sum_{k=j+1}^{\nu} \int_{\tau_{k-1}}^{\tau_k} e^{-A(s-t)}[Bu(s) + f]ds \\ &= - \sum_{k=j+1}^{\nu} \int_{\tau_{k-1}}^{\tau_k} e^{-A(s-t)}[Bu_k + f]ds \\ &= - \int_{\tau_j}^{\tau_\nu = \tau_1} e^{-A(s-t)}f\, ds - \sum_{k=j+1}^{\nu} \left(\int_{\tau_{k-1}}^{\tau_k} e^{-A(s-t)}Bds \right) \cdot u_k \\ &= d_j + \sum_{k=j+1}^{\nu} D_k u_k, \end{aligned} \tag{15}$$

the last equality serving to define

d_j and D_k, $k = j+1,\ldots,\nu$. The condition that $x(\tau_j)$ be in G is then

$$V_\ell \cdot (d_j + \sum_{k=j+1}^{\nu} D_k u_k) \leq C \ , \ \ell = 1,2,\ldots,L \tag{16}$$

(refer to (11)). This imposes linear restraints on the u_k, $k = j+1,\ldots,\nu$. In addition

$$u_{oi} \leq u_{ki} \leq u_{1i}, \tag{17}$$

denoting the component of u_k in question, for each k.

Performing (15) for each j, $j = 0,1,\ldots,\nu-1$ it is seen (16) and (17) describe $\nu(2n+L)$ linear constraints upon the vector U having νn components.

Thus there is a fixed vector Γ_ψ and a variable vector U. The problem now is to maximize $\Gamma_\psi \cdot U$, keeping U subject to the constraints defined by (16) and (17).

This problem may be succinctly stated as:
Maximize $\Gamma_\psi \cdot U$, subject to $AU \leq B$, where A is a matrix with dimensions $\nu(2n+L)$, νn and B is a vector of dimension $\nu(2n+L)$, the inequality being understood to hold between all components of the vectors involved. This is called the General Maximization Problem of Linear Programming. Thus any further effort in this approach must be directed toward finding the best way to solve this linear programming problem. It is observed that the matrix whose entries are

$$V_\ell \cdot \sum_{k=j+1}^{\nu} D_k u_k, \ \ell = 1,2,\ldots,L, \text{ and } j = 0,1,\ldots,\nu-1 \text{ has}$$

a certain "block triangular" character. This could conceivably simplify the linear programming.

It is also noted that the constraints on U are independent of the choice of ψ so that the constraint set is fixed once and for all. Because a large number of ψ will undoubtedly be used, attention should be directed to the possibility that the U_1 selected for one ψ_1 could be used as an initial guess in some sort of an iteration process for finding U_2 corresponding to a ψ_2 whose initial conditions were near those of ψ_1.

CONCLUSIONS

A modified form of the maximum principle is seen to lead to a formally correct "backing out" procedure. The effectiveness of this procedure rests on linear programming.

REFERENCES

1. Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., "The Mathematical Theory of Optimal Processes", Interscience Publishers, New York, 1962.
2. Dunford, N. and Schwartz, J., "Linear Operators", Interscience Publishers, New York, 1958.